REFERENCES

1. CARSLOW H. and JAEGER D., Thermal Conductivity of Solids. Moscow, NAUKA, 1964.
2. PARKUS G., Unsteady Temperature Stresses. Moscow, FIZMATGIZ, 1963.
3. JAHNKE E., EMDE F. and LÖSCHE F., Special Functions, Moscow, NAUKA, 1977.
4. GUZ' A.N., Principles of the Theory of Stability of Mine Shafts. Kiev. NAUKOVA DUMKA, 1977.
5. ALEKSANDROV A.Ia. and SOLOV'EV Iu.I., Three-dimensional Problems of the Theory of Elasticity, Moscow, NAUKA, 1978.

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AXISYMMETRIC CONTACT PROBLEMS FOR NON-UNIFORMLY AGING LAYERED VISCOELASTIC FOUNDATIONS*

A.V. MANZHIROV

Contact problems for non-uniformly aging multilayered viscoelastic foundations are studied. It is assumed that the thickness of the top layer is much less than the characteristic dimension of the area of contact. Integral equations of mixed problems containing Fredholm and Volterra operators are derived and a method for solving them is given. Basic versions of the non-uniform aging of a packet of layers are studied and the case in question is compared analytically with the classical case. Numerical computations of the characteristic parameters are given.

1. We consider the contact problems of the frictionless impression of a rigid circular stamp, using a constant force P, into a multilayered non-uniformly aging viscoelastic foundation consisting of:

1) a thin non-uniformly aging layer and a uniformly aging layer of arbitrary thickness, without friction between the layers;

2) a thin layer, a non-uniformly aging core foundation and a uniformly aging layer, with the first two layers coupled to each other, and resting on the third;

3) the packet is composed of the layers listed in 2), with zero friction between them.

We shall call the layer thin if the characteristic dimension of the part of the layer subjected to the active load is much greater than its thickness. The layer thicknesses are h, l and H respectively. Smooth contact or coupling with the non-deformable support occurs at the lower edge of the multilayer packet. The surface of the stamp base is described by the function $g(r)$, and the region of contact by the inequality $r \leqslant a$.

We write the equations of state of the layer materials in the form /1/

$$e_{ij}(t, r, z) = (1+\nu)\left[\frac{S_{ij}(t, r, z)}{E} - \int_{\tau_0}^{t} \frac{S_{ij}(\tau, r, z)}{E} K(t + \varkappa(z), \tau + \varkappa(z))\, d\tau\right]$$

$$\varepsilon(t, r, z) = (1-2\nu)\left[\frac{\sigma(t, r, z)}{E} - \int_{\tau_0}^{t} \frac{\sigma(\tau, r, z)}{E} K(t + \varkappa(z), \tau + \varkappa(z))\, d\tau\right]$$

$$K(t, \tau) = E\frac{\partial}{\partial \tau} C(t, \tau)$$

Here $e_{ij}(t, r, z)$ and $S_{ij}(t, r, z)$ are the deviators of the strain and stress tensors, respectively, $3\varepsilon(t, r, z)$ is the volume strain, $\sigma(t, r, z)$ is the mean hydrostatic pressure, $K(t, \tau)$ is the tensile creep kernel, $C(t, \tau)$ is a measure of the creep, r and z are cylindrical coordinates of a point of the body, τ_0 is the time of application of the load, $\varkappa(z)$ is the non-uniform aging function, and E and ν denote the instantaneous modulus of elasticity and

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Poisson's ratio, which are both constant.

Below, we shall use the notation that is standard for a cylindrical system of coordinates and denote the quantities referring to particular layers by the following superscripts: the thin layer by 1, the core layer by c, and the uniformly aging layer by 2.

If $h/(2a) \ll 1$, then we can show that when a normal load $q(r, t)$ acts on the packet of layers, we have

$$\sigma_z^1 = q(r, t), \quad \varepsilon_z^1 = \partial w_1/\partial z \tag{1.1}$$

$$\varepsilon_z^1 = (2\theta_1)^{-1}\left[q(r, t) - \int_{\tau_0}^{t} q(r, \tau) K_1(t + \varkappa(z), \tau + \varkappa(z))\, d\tau\right]$$

where, in the case of zero friction at the lower boundary of the thin layer we have

$$\theta_1 = \frac{E_1}{2(1 - \nu_1^2)} \tag{1.2}$$

and in the case of coupling with the core foundation

$$\theta_1 = \frac{E_1(1 - \nu_1)}{2(1 - \nu_1 - 2\nu_1^2)} \tag{1.3}$$

$$\sigma_z^c = q(r, t) \tag{1.4}$$

$$\varepsilon_z^c = (2\theta_c)^{-1}\left[q(r, t) - \int_{\tau_0}^{t} q(r, \tau) K_c(t + \varkappa_c(z), \ \tau + \varkappa_c(z))\, d\tau\right]$$

$$\varepsilon_z^c = \frac{\partial w_c}{\partial z}, \quad \theta_c = \frac{E_c(1 - \nu_c)}{2(1 - \nu_c - 2\nu_c^2)}$$

Since the normal stresses σ_z^1 and σ_z^c do not vary across the layer thickness (see (1.1), (1.4)) and the lower layer ages uniformly, we can apply the correspondence principle /2/ and obtain, by virtue of /3/,

$$w_2 = \frac{1}{\theta_2 H}\left[\int_0^a q(\rho, t)\, k\left(\frac{\rho}{H}, \frac{r}{H}\right)\rho\, d\rho - \right. \tag{1.5}$$

$$\left. \int_{\tau_0}^{t}\int_0^a q(\rho, \tau)\, k\left(\frac{\rho}{H}, \frac{r}{H}\right)\rho\, d\rho\, K_2(t - \tau_2, \tau - \tau_2)\, d\tau\right]$$

$$\theta_2 = \frac{E_2}{2(1 - \nu_2^2)}$$

$$k(t, \tau) = \int_0^\infty L(u)\, J_0(tu)\, J_0(\tau u)\, du \tag{1.6}$$

where w_2 denote the vertical displacements of the points of the lower layer and τ_2 is the instant it occurs; in the case when the lower edge of the multilayer packet is coupled to the non-deformable support, we have

$$L(u) = \frac{2\varkappa\, \mathrm{sh}\, 2u - 4u}{2\varkappa\, \mathrm{ch}\, 2u + 4u^2 + 1 + \varkappa^2}, \quad \varkappa = 3 - 4\nu_2 \tag{1.7}$$

and in the case of a smooth contact

$$L(u) = \frac{\mathrm{ch}\, 2u - 1}{\mathrm{sh}\, 2u + 2u} \tag{1.8}$$

Remembering that by virtue of the condition of contact under the stamp $w_1 = \delta(t) - g(r)$, and matching the displacements along the lines of interlayer coupling, we obtain from (1.1)–(1.5) the following integral equations for the problems posed ($\delta(t)$ is the rigid displacement of the stamp)

$$\frac{h}{2\theta_1}\left[q(r, t) - \int_{\tau_0}^{t} q(r, \tau) h^{-1}\int_0^h K_1(t + \varkappa(z), \ \tau + \varkappa(z))\, dz\, d\tau\right] + \frac{1}{H\theta_2}\left[\int_0^a q(\rho, t)\, k\left(\frac{\rho}{H}, \frac{r}{H}\right)\rho\, d\rho - \right. \tag{1.9}$$

$$\left. \int_{\tau_0}^{t}\int_0^a q(\rho, \tau)\, k\left(\frac{\rho}{H}, \frac{r}{H}\right)\rho\, d\rho K_2(t - \tau_2, \ \tau - \tau_2)\, d\tau\right] + \Lambda = \ \delta(t) - g(r) \quad (r \leqslant a)$$

corresponding to the following cases:

1) $$\Lambda = 0, \quad \theta_1 = \frac{E_1}{2(1-\nu_1^2)}$$

2) $$\Lambda = \frac{l}{2\theta_c}\left[q(r,t) - \int_{\tau_0}^{t} q(r,\tau)\, l^{-1}\int_0^l K_c(t+\varkappa_c(z),\ \tau+\varkappa_c(z))\,dz\,d\tau\right]$$

$$\theta_1 = \frac{E_1(1-\nu_1)}{2(1-\nu_1-2\nu_1^2)}, \quad \theta_c = \frac{E_c(1-\nu_c)}{2(1-\nu_c-2\nu_c^2)}$$

3) the same as in 2), but with

$$\theta_1 = \frac{E_1}{2(1-\nu_1^2)}$$

The above relations must be supplemented by the static condition

$$P = 2\pi\int_0^a q(\rho,t)\,\rho\,d\rho \tag{1.10}$$

We will now dwell on some of the properties of the creep measure, the creep kernels and the relaxation. According to /1,2/ we have

$$\begin{gathered} C(t,t) = 0; \quad \lim_{t\to\infty}\frac{\partial C(t,\tau)}{\partial t} = 0, \ \forall\tau \\ \lim_{t\to\infty} C(t,\tau) = \varphi(\tau), \quad \forall\tau; \quad \lim_{\tau\to\infty}\varphi(\tau) = C_0 \\ C(t,\tau) = C(t-\tau,\tau) = \varphi(\tau)\,f(t-\tau) \end{gathered} \tag{1.11}$$

where $\varphi(\tau)$ is a function reflecting the aging process of the material and $f(t-\tau)$ characterizes its hereditary properties.

In addition, ($R(t,\tau)$ is the resolvent of the kernel $K(t,\tau)$)

$$\int_{\tau_0}^{t} R(t,\tau)\,d\tau, \quad \int_{\tau_0}^{t} K(t,\tau)\,d\tau$$

are continuous and bounded functions.

We shall assume that the hereditary properties of the layer materials are the same, in which case the equations of contact problems can be reduced to a single form.

2. Let us construct the solutions of the integral equations of the axisymmetric contact problems (1.9). Since they are mathematically equivalent, we shall consider, to be specific Eq.(1.9) for case 1) with conditions (1.10). Taking into account the notation

$$\begin{gathered} r^* = ra^{-1}, \quad \rho^* = \rho a^{-1}, \quad t^* = t\tau_0^{-1}, \quad \tau^* = \tau\tau_0^{-1} \\ \varkappa^*(z) = \varkappa(z)\,\tau_0^{-1}, \quad c = {}^1\!/_2 ha^{-1}\theta_2\theta_1^{-1} \\ q^*(r^*,t^*) = q(r,t)\,\theta_2^{-1}, \quad \delta^*(t^*) = \delta(t)\,a^{-1}, \quad g^*(r^*) = g(r)\,a^{-1} \\ \lambda = Ha^{-1}, \quad P^* = (2\pi a^2\theta_2)^{-1}P, \quad k^*\left(\frac{r^*}{\lambda}, \frac{\rho^*}{\lambda}\right) = k\left(\frac{r}{H}, \frac{\rho}{H}\right)\lambda^{-1} \\ E_iC_i(t,\tau) = C_i^*(t^*,\tau^*), \quad K_i^*(t^*,\tau^*) = \frac{\partial}{\partial\tau^*}C_i^*(t^*,\tau^*) \\ \theta_i = \frac{E_i}{2(1-\nu_i^2)} \quad (i = 1, 2) \end{gathered}$$

and henceforth omitting, the asterisks, we obtain

$$\begin{gathered} c\left[q(r,t) - \int_1^t q(r,\tau)\,K_1^\circ(t,\tau)\,d\tau\right] + \int_0^1 q(\rho,t)\,k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)\rho\,d\rho - \\ \int_1^t\int_0^1 q(\rho,\tau)\,k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)\rho\,d\rho\,K_2(t-\tau_2, \tau-\tau_2)\,d\tau = \\ \delta(t) - g(r) \quad (r \leqslant a,\ 1 \leqslant t \leqslant T < \infty) \\ K_1^\circ(t,\tau) = h^{-1}\int_0^h K_1(t+\varkappa(z),\ \tau+\varkappa(z))\,dz \end{gathered} \tag{2.1}$$

$$P=\int_0^1 q(\rho,t)\rho\,d\rho \tag{2.2}$$

In accordance with /4,5/ we consider the integral equation equivalent to (2.1)

$$\Phi q\equiv c\left[q(r,t)-q(r,1)-\int_1^t q(r,\tau)K_1^\circ(t,\tau)\,d\tau\right]+\int_0^1[q(\rho,t)-q(\rho,1)]\,k\left(\frac{\rho}{\lambda},\frac{r}{\lambda}\right)\rho\,d\rho- \tag{2.3}$$

$$\int_1^t\int_0^1 q(\rho,\tau)\,k\left(\frac{\rho}{\lambda},\frac{r}{\lambda}\right)\rho\,d\rho K_2(t-\tau_2,\ \tau-\tau_2)\,d\tau\ =\delta(t)-\delta(1)\quad (r\leqslant a,\ 1\leqslant t\leqslant T<\infty)$$

We shall seek the solution of (2.3) in the form

$$q(r,t)=q_0(r)+q_1(r,t),\quad \int_0^1 q_1(\rho,t)\rho\,d\rho=0 \tag{2.4}$$

$$\delta(t)=\delta y(t)+\delta_0+\sum_{i=1}^\infty \delta_i y_i(t) \tag{2.5}$$

Substituting (2.4), (2.5) into (2.3), we obtain

$$y(t)=C_1^\circ(t,1),\ \ F=\varphi_2(1-\tau_2)\,\varphi_1^\circ(1)^{-1} \tag{2.6}$$

$$C_1^\circ(t,\tau)=h^{-1}\int_0^h C_1(t+\varkappa(z),\quad \tau+\varkappa(z))\,dz=\ h^{-1}\int_0^h\varphi_1(\tau+\varkappa(z))\,dzf(t-\tau)=\varphi_1^\circ(\tau)f(t-\tau),$$

$$C_2(t,\tau)=\varphi_2(\tau)f(t-\tau)$$

$$cq_0(r)+F\int_0^1 q_0(\rho)\,k\left(\frac{\rho}{\lambda},\frac{r}{\lambda}\right)\rho\,d\rho=\delta \tag{2.7}$$

$$P=\int_0^1 q_0(\rho)\rho\,d\rho$$

$$\Phi q_1=\sum_{i=1}^\infty\delta_i[y_i(t)-y_i(1)] \tag{2.8}$$

Let us investigate Eq.(2.7).

Theorem 1. The operator A

$$\left(A\varphi=\int_0^1\varphi(\rho)\,k\left(\frac{\rho}{\lambda},\frac{r}{\lambda}\right)\rho\,d\rho\right)$$

is completely continuous selfconjugate and positive definite from $L_2(\Omega)$ into $L_2(\Omega)$ where Ω is the unit circle. The proof is easy, and we shall merely note that

$$\int_0^1\int_0^1 k^2\left(\frac{\rho}{\lambda},\frac{r}{\lambda}\right)\rho r\,d\rho\,dr=K^2<\infty\quad(\lambda\in(0,\infty)) \tag{2.9}$$

Theorem 2. A solution of Eq.(2.7) exists in the space $L_2(\Omega)$ and is unique for any value of $c,\lambda\in(0,\infty)$. The proof is obvious from Theorem 1 and the postiveness of F.

We shall seek the solution of (2.7) in the form of a series in terms of a complete, orthonormalized in $L_2(\Omega)$ system of polynomials $P_m^*(r)$ ($P_m(x)$ is the Legendre polynomial)

$$P_m^*(r)=\sqrt{4m+2}\,P_m(1-2r^2)\quad(m=0,1,2\ldots) \tag{2.10}$$

$$\int_0^1 P_m^*(r)\,r\,dr=\begin{cases}2^{-1/2}, & m=0\\ 0, & m\neq 0\end{cases}$$

Infact, let

$$q_0(r)=\sum_{j=0}^\infty D_jP_j^*(r)=\frac{\delta}{\sqrt{2}}\sum_{j=0}^\infty d_jP_j^*(r) \tag{2.11}$$

Writing the kernel of the integral equation in the form of a double series in terms of the polynomial system chosen

$$k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn}(\lambda) P_m^*(r) P_n^*(\rho) \tag{2.12}$$

where, using the tabulated integral /6/, we have,

$$r_{mn}(\lambda)=2\left[(2m+1)(2n+1)\right]^{1/2}\lambda\int_0^{\infty}\frac{L(u)}{u^2}J_{2m+1}\left(\frac{u}{\lambda}\right)J_{2n+1}\left(\frac{u}{\lambda}\right)du$$

and substituting (2.11), (2.12) into (2.7) we obtain (δ_{0n} is the Kronecker delta)

$$cd_n+F\sum_{j=0}^{\infty} d_j r_{jn}(\lambda)=\delta_{0n},\quad \delta=2Pd_0^{-1} \tag{2.13}$$

By virtue of the Parseval relation, (2.9) and (2.12), we obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn}^2(\lambda)<\infty \tag{2.14}$$

i.e. the operator on the left side of (2.13) is completely continuous, selfconjugate and positive definite from l_2 into l_2. Solution (2.13) exists and is unique for all $c, \lambda\in(0,\infty)$ and can be found using the reduction method /7/.

Let us now consider Eq.(2.8). We introduce a space of functions square integrable in Ω, whose surface integral is equal to zero. We denote this space by $L_2^\circ(\Omega)$.

Theorem 3. The space $L_2^\circ(\Omega)$ is a complete Hilbert space, any function of which can be written in terms of a series in a chosen system of polynomials, beginning with the first.

Let $\{f_n\}$ be a fundamental sequence from $L_2^\circ(\Omega)$. Since $L_2^\circ(\Omega)\subset L_2(\Omega)$, therefore $\{f_n\}$ converges at least to $f\in L_2(\Omega)$, i.e.

$$\|f-f_n\|_{L_2(\Omega)}<\varepsilon,\quad n>N \tag{2.15}$$

We shall show that $f\in L_2^\circ(\Omega)$. In fact,

$$\left|\int_\Omega f\,d\omega\right|=\left|\int_\Omega (f-f_n)\,d\omega+\int_\Omega f_n\,d\omega\right|\leqslant\int_\Omega|f-f_n|\,d\omega\leqslant\sqrt{\pi}\,\|f-f_n\|_{L_2(\Omega)}$$

and this, with (2.15) and the fact that f is independent of n, yields

$$\int_\Omega f\,d\omega=0$$

which proves the completeness of $L_2^\circ(\Omega)$. The last assertion of the theorem follows from the properties of the polynomials (2.10).

Theorem 4. The operator B

$$\left(B\varphi=\int_0^1\varphi(\rho)k_1(\rho,r,\lambda)\rho\,d\rho,\right.$$

$$\left.k_1(\rho,r,\lambda)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} r_{mn}(\lambda)P_m^*(r)P_n^*(\rho)\right)$$

is completely continuous, selfconjugate, positive definite and acts from $L_2^\circ(\Omega)$ into $L_2^\circ(\Omega)$.

The theorem can be proved using the obvious estimate (see (2.14))

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} r_{mn}^2(\lambda)<\infty$$

Let β_i be the eigenvalues of the operator A, and α_i the eigenvalues of B. We can assert that /8/

$$\beta_{i+1}^{-1}<\alpha_i^{-1}<\beta_i^{-1} \tag{2.16}$$

Moreover, according to Theorem 4 and /9/, the eigenfunctions ψ_i of the operator B corresponding to various eigenvalues α_i, form a complete orthogonal system in $L_2^\circ(\Omega)$.

Let us construct this system. We write the eigenfunctions in the form (see Theorem 3)

$$\psi_i(r)=\sum_{j=1}^{\infty} a_j^i P_j^*(r) \tag{2.17}$$

Then α_i are given by

$$\sum_{j=1}^{\infty} r_{jn} a_j^i = \alpha_i^{-1} a_n^i \quad (i, n=1, 2\ldots) \tag{2.18}$$

Substituting α_i into the equation

$$\alpha_i \sum_{j=0}^{\infty} r_{nj}(\lambda) a_j^i = a_n^i + \delta_{0n} \quad (n=0, 1, \ldots; \ i=1, 2, \ldots) \tag{2.19}$$

we obtain $\{a_j^i\} \in l_2$ $(j = 0,1, \ldots)$, which can always be done by virtue of (2.16). We note that $\{a_j^i\}$ $(j = 1, 2\ldots)$ will satisfy relation (2.18), i.e. will yield the coefficients of the expansions of the eigenfunctions. In fact $a_0^i = \Delta^{-1}\Delta_1$ where Δ is the fundamental determinant of the system (2.19) and Δ_1 is the auxilliary determinant obtained from the previous one by replacing the first column by the elements $\{1, 0, \ldots, 0, \ldots\}$. But Δ_1 is the determinant of system (2.18), therefore $\Delta_1 = 0$ and $a_0^i = 0$ $(i = 1, 2, \ldots)$.

The eigenfunctions of the operator B obtained have the following property (see (2.19)):

$$\alpha_i A\psi_i = \psi_i + \sqrt{2} \quad (i=1, 2, \ldots) \tag{2.20}$$

To facilitate the subsequent transformations we write

$$q_i(r) = \frac{\alpha_i \delta_i}{\sqrt{2}} \psi_i(r) \tag{2.21}$$

We shall seek the solution of (2.8) in the form of a series in terms of a complete, orthogonal in $L_2^\circ(\Omega)$ system of functions $\{q_i(r)\}$, i.e.

$$q_1(r, t) = \sum_{i=1}^{\infty} z_i(t) q_i(r) \tag{2.22}$$

The system $\{q_i(r)\}$, constructed using formulas (2.17)—(2.21) has the following property:

$$\alpha_i \int_0^1 q_i(\rho)\, k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) \rho\, d\rho = q_i(r) + \alpha_i \delta_i \tag{2.23}$$

Substituting (2.22) and (2.23) into (2.8) and equating the coefficients of $q_i(r)$ and δ_i, we obtain

$$z_i(1) = z_i(t) - \int_1^t z_i(\tau) K_i(t, \tau)\, d\tau \tag{2.24}$$

$$K_i(t, \tau) = \frac{K_2(t-\tau_2, \tau-\tau_2) + \alpha_i c K_1^\circ(t, \tau)}{1+\alpha_i c}$$

$$y_i(t) = z_i(t) - \int_1^t z_i(\tau) K_2(t-\tau_2, \ \tau-\tau_2)\, d\tau \tag{2.25}$$

The sequence $\{z_i(1)\}$ is obtained form the obvious relation

$$q(r, 1) = q_0(r) + \sum_{i=1}^{\infty} z_i(1) q_i(r) \tag{2.26}$$

In facts, the function $q_0(r)$ has already been determined and $q(r, 1)$ can be found in exactly the same manner from (2.1) for $t = 1$, and if $g(r) \in L_2(\Omega)$, then

$$q(r, 1) = \sum_{j=0}^{\infty} B_j P_j^*(r) \tag{2.27}$$

$$cB_n + \sum_{m=0}^{\infty} r_{mn}(\lambda) B_m = \frac{\delta(1)}{\sqrt{2}} \delta_{0n} - g_n$$

$$P = \frac{B_0}{\sqrt{2}} (n=0, 1, \ldots), \ g(r) = \sum_{k=0}^{\infty} g_k P_k^*(r)$$

Substituting (2.11), (2.17), (2.27), into (2.26) we obtain

$$B_0 = D_0 \tag{2.28}$$

$$\sum_{i=1}^{\infty} \alpha_i \delta_i z_i(1) a_j^i = \sqrt{2}(B_j - D_j) \quad (j = 1, 2, \ldots)$$

Suppose now that $\delta_i = \alpha_i^{-2}$, and let us investigate the operator on the left side of (2.28). We write

$$D = \| \alpha_i^{-1} a_j^i \|, \quad \mathbf{z} = \{z_i(1)\}, \quad \mathbf{b} = \{\sqrt{2}(B_j - D_j)\}$$

Then (2.28) takes the form

$$D\mathbf{z} = \mathbf{b} \tag{2.29}$$

Theorem 5. The operator D is completely continuous and acts from l_2 into l_2. In fact,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} [\alpha_i^{-1} a_j^i]^2 = \sum_{i=1}^{\infty} \alpha_i^{-2} \sum_{j=1}^{\infty} (a_j^i)^2 < \infty$$

since $\{a_j^i\} \in l_2$ (see (2.19)) and by Shur's theorem /10/

$$\sum_{i=1}^{\infty} \alpha_i^{-2} < \infty$$

Theorem 6. The operator D^{-1} exists. We note that the columns of the matrix of the operator D contain the coefficients of the expansion of the linearly independent eigenfunctions $q_i(r)$, and by virtue of the positive definiteness of B, all $\alpha_i^{-1} > 0$. Then the matrix columns are linearly independent and its determinant is therefore non-zero. Using Theorems 5 and 6 we can conclude that an inverse of D exists but is unbounded. Applying the method of reguarlization /11/ we obtain the approximate solution of (2.29), and

$$\sum_{i=1}^{\infty} z_i^2(1) < \infty$$

Substituting $z_i(1)$ into (2.24) we obtain

$$z_i(t) = z_i(1) \left[1 + \int_1^t R_i(t, \tau) \, d\tau \right]$$

where $R_i(t, \tau)$ is the resolvent of the kernel $K_i(t, \tau)$, and $y_i(t)$ are found from (2.25). Finally, making use of relations (2.5), (2.6) we obtain

$$\delta_0 = \delta(1) - \sum_{i=1}^{\infty} \alpha_i^{-2} y_i(1)$$

Theorem 7. The series

$$\sum_{i=1}^{\infty} z_i(t) q_i(r)$$

converges uniformly in $L_2^\circ(\Omega)$ in $t \in [1, T]$ $(T < \infty)$ and defines there a function that is continuous in time.

Theorem 8. The series

$$\sum_{i=1}^{\infty} \alpha_i^{-2} y_i(t)$$

converges in $C[1, T]$ uniformly in $t \in [1, T]$ to the function belonging to this space.

Omitting the detailed proofs, we merely note that the functions are constructed with the help of Shur's theorem (see Theorem 5), estimate (2.30) and the properties of the rheological characteristics of the medium: (1.11), etc.

Thus we have obtained the functions of contact stresses continuous in t in $L_2(\Omega)$, and a time-continuous settling function, both satisfying the integral equation of the problem in question.

3. Let us consider the basic cases of non-uniform aging of a two-layer foundation. Let

$$\varphi_1(\tau + \varkappa(z)) = C_0 + A_0 e^{-\beta(\tau + \varkappa(z))}$$

Then by (2.6) we have

$$\varphi_1^\circ(\tau) = C_0 + A_0 \mu e^{-\beta\tau}, \quad \mu = h^{-1} \int_0^h e^{-\beta\varkappa(z)} \, dz \tag{3.1}$$

The non-uniform aging parameter μ and the instant of the preparation of the lower layer τ_2 together completley determine the non-uniform aging of a packet of layers.

The case of natural aging of a layer occurs when the growth of the elements of its lower edge is greatest, and this corresponds to the process of erecting a top layer on the bottom layer. Then we can show that $-1 < \varkappa(z) \leqslant 0$, and by virtue of (3.1) $1 \leqslant \mu < e^{\beta}$.

The case of artificial aging of a layer will occur when the growth of the elements of its bottom edge is minimal. Indeed, it is reasonable to assume that the effect of external factors (irradiation, temperature, etc.) will affect precisely these elements least. Then $\varkappa(z) \geqslant 0$ and $0 < \mu \leqslant 1$.

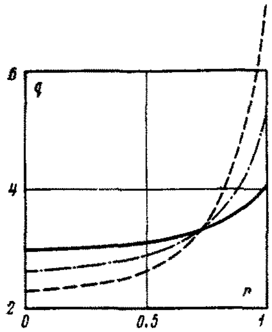

Fig.1

Let us investigate the limiting cases of a change in the non-uniform aging parameter μ.

Let $\mu = 1$. Then, provided that the layers are of the same material and $\tau_2 = 0$, the pressure distribution under the stamp will be the same as that in the analogous elastic problem. In fact we have here the case of uniform aging of a packet of layers when the creep does not affect the distribution of the contact stresses.

Let $\mu = 0$. Then the upper layer will work in accordance with the type of foundation whose model obeys the Volterra law of linear heredity /12/. The version $\mu = e^{\beta}$ corresponds to the case of piece-wise uniform aging of the foundation in question.

Note that if

$$f(t-\tau) = 1 - e^{-\gamma(t-\tau)} \tag{3.2}$$

then all functions of time appearing in the solution are obtained in explicit form, since we shall use the Arutiunian kernels /2/.

As an example consider the contact problem for a two-layer concrete packet lying without friction on a non-deformable support. We shall assume that the creep measure is given by expressions (3.1), (3.2), and

$$g(r) = 0;\ P = 1;\ \lambda = 6;\ c = 0,2;\ C_0 = 0{,}5522;\ A_0 = 4;\ \tau_2 = 0$$

1) The case of natural aging

$$\beta = 2.325;\ \gamma = 4.5;\ \tau_0 = 75 \text{ days};\ 1 < \mu \leqslant e^{\beta}$$

2) The case of artificial aging

$$\beta = 0.31;\ \gamma = 0.6;\ \tau_0 = 10 \text{ days};\ 0 < \mu \leqslant 1$$

Fig.1 shows the distribution of contact stresses relative to r, t and the parameter of non-uniform hardening μ; the dash-dot line refers to $t = 1$ and any value of μ (elastic solution), the solid line to $t = 2$, $\mu = 10$ (natural aging), and the dashed line to $t = 11$ $\mu = 0.1$ (artificial aging). Fig.2 shows the (solid lines) the relation between $q_{max}(t, \mu) = q(1, t, \mu)$, $q_{min}(t, \mu) = q(0, t, \mu)$ and μ for various fixed t for case 1). Curves $1-4$ correspond to $q_{max}(1,05, \mu)$, $q_{max}(2, \mu)$, $q_{min}(2, \mu)$, $q_{min}(1,05, \mu)$. It should be noted that the maximum (minimum) contact pressures decrease (increase) as the natural inhomogeneity increases (as μ increases).

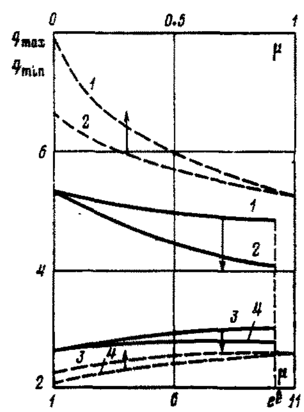

Fig.2

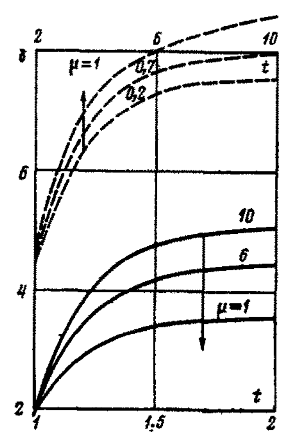

Fig.3

Analogous dashed lines in Fig.2 refer to case 2), and $q_{max}(11, \mu)$, $q_{max}(1.5, \mu)$, $q_{min}(1.5, \mu)$, $q_{min}(11, \mu)$ corresponds to curves $1-4$. We see that the maximum (minimum) contact stresses increase (decrease) as the artificial non-uniformity increases (as μ decreases from 1 to 0).

Fig.3 shows the relation $\delta(t)$ for fixed values of μ, for the cases of natural and forced aging (the solid and dashed lines, respectively). The function $\delta(t)$ increases with time t and tends to a limiting value, which is larger, the larger the parameter μ.

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REFERENCES

1. ARUTIUNIAN N.KH., Some problems of the theory of creep for non-uniformly aging bodies. Izv. Akad. Nauk SSSR, MTT, No.3, 1976.
2. ARUTIUNIAN N.KH., Some Problems of the Theory of Creep. Moscow-Leningrad, Gostekhizdat, 1952.
3. VOROVICH I.I., ALEKSANDROV V.M. and BABESHKO V.A., Non-classical Mixed Problems of the Theory of Elasticity. Moscow, NAUKA, 1974.
4. ALEKSANDROV V.M. and KOVALENKO E.V., On a class of integral equations of the mixed problems of mechanics of continuous media. Dokl. Akad. Nauk SSSR, Vol.252, No.2, 1980.
5. ALEKSANDROV V.M. and KOVALENKO E.V., The axisymmetric contact problem for a linearly deformable foundation of general type in the presence of wear. Izv. Akad. Nauk SSSR, MTT, No.5, 1978.
6. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series and Products, Moscow, Fizmatgiz, 1963.
7. KANTOROVICH L.V. and AKILOV G.P., Functional Analysis. Moscow, NAUKA, 1977.
8. BELLMAN R.E., Introduction of Matrix Analysis. N.Y. McGraw-Hill, 1960.
9. RIESZ F. and SZOKEFALVI-NAGY B., Functional Analysis. N.Y. Ungar, 1955.
10. ZABREIKO P.P., KOSHELEV A.I., KRASNOSEL'SKII M.A. et al. Integral Equations. Moscow, NAUKA, 1978.
11. IVANOV V.K., VASIN V.V. and TANANA V.P., The Theory of Linear Ill-Posed Problems and its Applications. Moscow, NAUKA, 1978.
12. ROBOTNOV Iu.N., Elements of the Hereditary Mechanics of Solids. Moscow, NAUKA, 1977.

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STEADY STATE BOUNDARY FLOWS IN THE LIGHT OF THE GENERALIZED KARMAN THEORY*

V.V. NOVOZHILOV

Results are given, based on the generalization in /1/ of the Karman theory of turbulence, obtained within the last ten years. The advantages and disadvantages of the model of turbulent flows used are analyzed and comparisons are made with other models.

1. Blasius's empirical formula of (1911) represents the first significant success in the applied theory of turbulence

$$\lambda \operatorname{Re}^{1/4} = 0{,}316 \tag{1.1}$$

The formula expresses the dependence of the coefficient of resistance on the Reynolds number in steady state flow in a straight pipe of circular cross-section. However, the relationship had no connection with the Reynolds equation and was therefore considered to represent an achievement in hydraulics rather than hydrodynamics. In the 1920-s Prandtl proposed, while developing the Reynolds' and Bussinesq's ideas, the phenomenological theory of turbulent steady-state flows, i.e. the mixing-length theory.

In fact, the problem was that of constructing a model of a non-linearly viscous fluid the laminar flow of which would be identical (in velocity profiles and stress distribution) with the averaged turbulent flow (with analogous boundary conditions). Prandtl's idea was complemented by Karman who put forward the idea of the selfsimilarity of steady-state turbulent flows. As a result a solution was obtained for the averaged turbulent flow in a straight pipe of circular cross-section, as well as results for the velocity profiles, and the relation $\lambda = f(\mathrm{Re})$, which agreed well with experimental data. The latter relation was practically

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